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Implicit Euler and Lie splitting discretizations of nonlinear parabolic equations with delay

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Abstract A convergence analysis is presented for the implicit Euler and Lie splitting schemes when applied to nonlinear parabolic equations with delay. More precisely, we consider a vector field which is the sum of an unbounded dissipative operator and a delay term, where both point delays and distributed delays fit into the framework. Such equations are frequently encountered, e.g. in population dynamics. The main theoretical result is that both schemes converge with an order (of at least) $q = 1/2$, without any artificial regularity assumptions. We discuss implementation details for the methods, and the convergence results are verified by numerical experiments demonstrating both the correct order, as well as the efficiency gain of Lie splitting as compared to the implicit Euler scheme.

Keywords Nonlinear parabolic equations · Delay differential equations · Convergence orders · Implicit Euler · Lie splitting

Mathematics Subject Classification (2000) 65L03 · 65J08 · 65M15

1 Introduction

Delay differential equations (DDEs) are equations which depend on the system state at previous times. They have important applications in engineering and science. As a prototypical example, we consider a model from population dynamics: The density $u(t,x)$ of a population at time t and location x in some habitat is modeled by the

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following parabolic DDE:

$$\frac{\partial}{\partial t}u(t,x) = \nabla\cdot\big(D\nabla u(t,x)\big)+g\big(u(t-1,x)\big). \tag{1.1}$$

The first term on the right-hand side represents the dispersal of the population throughout the habitat and the second term models the increase due to births. Here, the delayed argument $t-1$ takes into account the average gestation period of the population and the nonlinear function g models the population growth with respect to the population density. Surveys of delay-dependent population dynamics can be found in [11, 13].

The diffusion constant D is often a function of the population density, e.g. compare with the experimental study [14] on insect dispersal. This implies that the solution of (1.1) typically lacks higher order regularity in both time and space. To illustrate these features let $D(u) = |\nabla u|^{r-2}$ and $g = 0$. Then equation (1.1) reduces to the evolution of the r-Laplacian, and for suitable initial data the solution is given by the closed expression [2] (see also [9, p.160]):

$$u(t,x) = \frac{1}{(t+1)^{\lambda}}\left[1-\kappa\Big(\frac{|x|}{(t+1)^{\lambda}}\Big)^{r/(r-1)}\right]_{+}^{(r-1)/(r-2)},$$

where $\lambda = 1/(2r-2)$, $\kappa = \lambda^{1/(r-1)}(r-2)/r$ and $[\cdot]_{+} = \max\{\cdot,0\}$. Note that the solution is not continuously differentiable and has compact support for all times $t\geq 0$, i.e. the propagation speed is finite, in contrast to the linear case. We refer to [9, 15] for further details regarding nonlinear parabolic equations.

Due to the lack of time-regularity it is not, in general, possible to prove that a time discretization of the problem converges with an order greater than $q = 1$. Furthermore, since there is a diffusion term present, the spatial discretization of the equation will result in a stiff ODE system and therefore requires the usage of implicit schemes. Of the few remaining numerical methods the implicit Euler scheme is then the natural choice, but it is often computationally costly. An alternative is given by splitting methods, where the flows related to the diffusion and delay terms are approximated separately. This can dramatically reduce the computational cost.

The aim of this paper is to derive a new convergence analysis for the implicit Euler and Lie splitting schemes when applied to fully nonlinear equations of the same structure as (1.1). Our approach is based on rewriting the parabolic DDE as an abstract evolution equation in the spirit of [18] and then deriving a convergence analysis by employing the theory of dissipative operators [1]. The schemes are readily applicable to the abstract evolution equation, but quite some care needs to be taken when discretizing the concrete parabolic DDE. We therefore also discuss the implementation details.

In the abstract setting of dissipative evolution equations, a convergence order $q = 1/2$ has been established in [8] for the implicit Euler scheme. This convergence rate is in fact optimal for general dissipative vector fields, as observed by [16]. For general time stepping schemes a Lax-type theorem is provided by [7], and a related result [6] yields convergence of the Lie splitting under rather weak assumption on the abstract equation. In the setting of linear evolution equations, convergence of order $q = 1$ for the Lie splitting may be proved using the techniques in [10]. In the DDE case, this has been done in [4], where convergence orders are also proved in the semilinear case under additional structural assumptions on the delay term and on the initial data. Finally, a general survey of other approaches and numerical methods for DDEs can be found in [5] and the references therein. See also [3,12].

An outline of our paper is as follows: In Section 2 we describe the problem class, discuss the reformulation to an abstract evolution equation, define the two schemes and discuss implementation details. In Section 3 we recapitulate some general theory of dissipative operators and give the precise assumptions on the problem class. What these assumptions mean for the abstract evolution setting is discussed in Section 4, which also summarizes some basic results. The convergence analysis is given in Section 5 and we conclude with two numerical experiments which confirm our theoretical results in Section 6.

2 Time stepping schemes

Consider the equation

$$\dot{u}(t) = fu(t) + g\Phi u_t, \quad (u(0)\,;u_0) = (\zeta(0)\,;\zeta),$$

where $u(t)$ belongs to the Hilbert space H, and the history segment $u_t : [-1,0] \to H$ is defined by $u_t(\sigma) = u(t+\sigma)$. The unbounded operator f is assumed to be dissipative, as described later, and the delay operator is assumed to be an integral of the form

$$\Phi\rho = \int_{-1}^{0} \rho(\sigma)\mathrm{d}\eta(\sigma),$$

with $\eta : [-1,0] \to \mathbb{R}$ having bounded variation. Delay operators on this form include *distributed* delays such as $\Phi\rho = \int_{-1}^{0} \rho(\sigma)\mathrm{d}\sigma$ and *point delays* such as $\Phi\rho = \rho(-1)$.

In order to handle both the history dependency and the inherent low regularity, we reformulate the problem by introducing the auxiliary equation

$$\dot{u}_t = \frac{\mathrm{d}}{\mathrm{d}\sigma} u_t, \quad u_t(0) = u(t).$$

With $U(t) = (u(t)\,;u_t)$ this yields the evolution equation

$$\dot{U} = (F+G)U, \quad U(0) = (\zeta(0)\,;\zeta),$$

where the operators F and G are given by

$$F = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 & g\Phi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix}.$$

It is now possible to employ classic implicit time stepping schemes, and the proposed separation of the vector field enables the use of splitting schemes. Due to the low regularity we consider the first order *implicit Euler* scheme, where a single time step is given by

$$(u_n\,;\rho_n) = (I - h(F+G))^{-1}(u_{n-1}\,;\rho_{n-1}),$$

and the *Lie splitting* scheme, given by

$$(v_n\,; \varphi_n) = (I - hF)^{-1}(I - hG)^{-1}(v_{n-1}\,; \varphi_{n-1}).$$

The initial conditions are the same in both schemes, $(u_0\,; \rho_0) = (v_0\,; \varphi_0) = (\zeta(0)\,; \zeta)$, and both u_n and v_n approximate $u(nh)$. These formal representations of the schemes are the key to the error analysis, but some care needs to be taken in order to implement them. We consider first the Lie splitting scheme.

As the action of $(I - hF)^{-1}$ reduces to the action of $(I - hf)^{-1}$ on the first component of the argument, we proceed to investigate the action of $(I - hG)^{-1}$. Let

$$(w\,; \psi) = (I - hG)^{-1}(u\,; \rho),$$

with $w = \psi(0)$. Then

$$w - hg\Phi\psi = u \quad \text{and} \tag{2.1}$$
$$\psi - h\psi' = \rho, \tag{2.2}$$

where we can solve (2.2) by using the integrating factor $\mathrm{e}^{-\sigma/h}$ and integrating from σ to 0. This yields

$$\psi(\sigma) = \mathrm{e}^{\sigma/h} w + \int_\sigma^0 \frac{1}{h}\mathrm{e}^{(\sigma - s)/h}\rho(s)\mathrm{d}s.$$

To shorten the notation, we introduce

$$\theta_h(\sigma) = \int_\sigma^0 \frac{1}{h}\mathrm{e}^{(\sigma - s)/h}\rho(s)\mathrm{d}s \quad \text{and} \quad k_h = \int_{-1}^0 \mathrm{e}^{\sigma/h}\mathrm{d}\eta(\sigma). \tag{2.3}$$

Using θ_h we can then express ψ by

$$\psi(\sigma) = \mathrm{e}^{\sigma/h} w + \theta_h(\sigma).$$

Since Φ is linear we have $\Phi\psi = k_h w + \Phi\theta_h$, and Equation (2.1) then becomes

$$w = u + hg\left(k_h w + \Phi\theta_h\right).$$

The computation of the n:th Lie step $(v_n\,; \varphi_n)$ can now be implemented as in Algorithm 1 and an illustration of the procedure is shown in Figure 2.1. We note that from a numerical point of view, it is only the first component v_n which is of practical interest; the history segment φ_n is only an auxiliary variable.

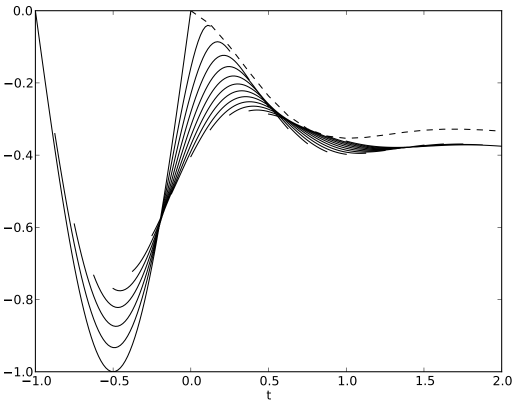

Fig. 2.1 The Lie splitting discretization of $\dot{u}(t) = -u(t) + u(t-1)$. The dashed line is the actual solution while the solid lines are the history segments. We note how the numerical error makes the approximations to $u(t)$ and $u_t(0)$ drift away from each other as the simulation progresses.

Algorithm 1 (A1)

Input: $(v_0\,; \varphi_0)$
1. Compute k_h;
2. **for** $j = 1 \ldots n$ **do**
3. Approximate $\theta_h(\sigma) = \int_\sigma^0 \frac{1}{h} e^{(\sigma-s)/h} \varphi_{j-1}(s) \mathrm{d}s$;
4. Compute $\Phi\theta_h$;
5. Approximate w by $w = v_{j-1} + hg\,(k_h w + \Phi\theta_h)$;
6. Compute $\psi(\sigma) = e^{\sigma/h} w + \theta_h(\sigma)$;
7. Update v_j by solving $(I - hf)v_j = w$;
8. Update $\varphi_j = \psi$;
9. **end for**

Output: $(v_n\,; \varphi_n)$

The steps (A1:3) and (A1:4) require some approximative representation of the function θ_h. Due to the low regularity in the problem, first-order interpolation is sufficient. The approximation required in (A1:5) can be done by fixed point iteration if the step size h is small enough, as we will assume that g is Lipschitz continuous with a moderate Lipschitz constant. Step (A1:7) typically requires the application of a spatial discretization of $I - hf$.

Consider now the implicit Euler scheme. If $(w\,; \psi) = (I - h(F+G))^{-1}(u\,; \rho)$ with $w = \psi(0)$, the same reasoning as above leads to

$$(I - hf)w = u + hg(k_h w + \Phi\theta_h) \quad \text{and} \quad \psi(\sigma) = e^{\sigma/h} w + \theta_h(\sigma).$$

Algorithm 2 summarizes the procedure for computing the implicit Euler approximation.

Algorithm 2 (A2)

Input: $(u_0\,;\rho_0)$
1. Compute k_h;
2. **for** $j = 1\ldots n$ **do**
3. Approximate $\theta_h(\sigma) = \int_\sigma^0 \frac{1}{h}\mathrm{e}^{(\sigma-s)/h}\rho_{j-1}(s)\mathrm{d}s$;
4. Compute $\Phi\theta_h$;
5. Approximate w by $(I-hf)w = u_{j-1} + hg(k_h w + \Phi\theta_h)$;
6. Compute $\rho(\sigma) = \mathrm{e}^{\sigma/h}w + \theta_h(\sigma)$;
7. Update $u_j = w$;
8. Update $\rho_j = \rho$;
9. **end for**

Output: $(u_n\,;\rho_n)$

This is very similar to Algorithm 1 except for (A2:5). Due to the presence of the unbounded f term it is not possible to use e.g. fixed point iteration here, while one may do so in (A1:5). This observation motivates the use of splitting in the semilinear case, since then also (A1:7) can be done very efficiently by tailored fast linear solvers for $(I-hf)v = w$. Further motivation for splitting arise from the case when g is a non-local operator, such as $g(u)(x) = \int_\Omega k(x,s)u(s)\mathrm{d}s$. Then (A2:5) increases in complexity for the implicit Euler scheme while the Lie splitting with fixed point iteration is largely unaffected.

3 Assumptions

We now state the precise assumptions on the operators f, g and Φ in the equation

$$\dot{u}(t) = fu(t) + g\Phi u_t, \quad (u(0)\,;u_0) = (\zeta(0)\,;\zeta). \tag{3.1}$$

First, we recall the notions of Lipschitz continuity and m-dissipativity. Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.

Definition 3.1 Let $E : \mathscr{D}(E) \subset X \to Y$ be an arbitrary operator. The *Lipschitz constant* $L_{X,Y}[E]$ is defined by

$$L_{X,Y}[E] = \sup_{\substack{u,v\in\mathscr{D}(E)\\ u\neq v}} \frac{\|Eu-Ev\|_Y}{\|u-v\|_X}.$$

We say that E is *Lipschitz continuous from* X *to* Y if $L_{X,Y}[E] < \infty$. If $X = Y$ we write $L_X[E] = L_{X,X}[E]$.

Definition 3.2 The operator $E : \mathscr{D}(E) \subset X \to X$ is m-dissipative if there exists a constant $M[E] \in [0,\infty)$ such that for all $h \in (0, 1/M[E])$ the operator $I-hE$ is surjective, i.e. $\mathscr{R}(I-hE) = X$, and

$$\|(I-hE)u-(I-hE)v\|_X \geq (1-hM[E])\|u-v\|_X, \quad \text{for all } u,v \in \mathscr{D}(E).$$

Assumption 1 *The operator $f : \mathscr{D}(f) \subset H \to H$ is m-dissipative with $M[f] = 0$. Furthermore, f is densely defined, i.e. $\overline{\mathscr{D}(f)} = H$.*

Example 3.1 Consider a bounded domain Ω with sufficiently regular boundary, and homogeneous Dirichlet boundary conditions. Then the *r-Laplacian* $\nabla \cdot (|\nabla u|^{r-2}\nabla u)$ with $r \geq 2$ on $L^2(\Omega)$ is m-dissipative. The *porous medium* operator $\Delta(|u|^{r-1}u)$ with $r \geq 1$ is m-dissipative on $H^{-1}(\Omega)$. See e.g. [1, Chapters 2 and 3], [15, Chapters 3 and 4] and [17, Chapter 10] for proofs of these assertions, as well as further examples.

We will let the delay operator Φ have one of two different forms. The first case handles *distributed* delays such as $\Phi\rho = \int_{-1}^{0} \rho(\sigma)\mathrm{d}\sigma$ while the second case treats *point delays* such as $\Phi\rho = \rho(-1)$. To be able to apply the same analysis, we will in both cases write the delay operator as the integral

$$\Phi\rho = \int_{-1}^{0} \rho(\sigma)\mathrm{d}\eta(\sigma),$$

where we place two different requirements on η.

Assumption 2

Case 1: The function $\eta : [-1,0] \to \mathbb{R}$ is given by

$$\eta(\sigma) = \int_{-1}^{\sigma} \xi(s)\mathrm{d}s,$$

where $\xi \in L^\infty(-1,0)$.

Case 2: The function $\eta : [-1,0] \to \mathbb{R}$ is of bounded variation, $\lim_{\sigma \to -1} \eta(\sigma) \neq 0$ *and $\eta(-1) = 0$.*

Example 3.2 In the first case, $\Phi\rho = \int_{-1}^{0} \xi(s)\rho(s)\mathrm{d}s$, and Φ is Lipschitz continuous from $L^p(-1,0;H)$ to H for all $1 \leq p < \infty$ with its Lipschitz constant bounded by $\|\xi\|_{L^{p/(p-1)}(-1,0)}$. In the second case, Φ is only defined on a subspace of $L^p(-1,0;H)$ and no longer Lipschitz continuous. An example of such a delay operator is the point delay $\Phi\rho = \rho(-1)$, given by taking $\eta = \chi_{(-1,0]}$, the characteristic function of the interval $(-1,0]$. Note, that for the sake of simplicity, we only consider real-valued functions η, but the theory can be extended to cover the case when $\eta(\sigma) : H \to H$ is a Lipschitz continuous operator [18, Section 4].

Assumption 3 *The operator $g : H \to H$ is Lipschitz continuous with Lipschitz constant $L_H[g] \leq 1$ and $g(0) = 0$.*

The properties $g(0) = 0$ and $L_H[g] \leq 1$ are only assumed for the sake of simplicity. A non-zero $g(0)$ can always be shifted into f. If $L_H[g] > 1$ then one can instead work with $\hat{g}u = g(u/L[g])$ if η is also rescaled as $\hat{\eta} = L[g]\eta$. None of these changes have any effect on the main result. A *locally* Lipschitz continuous g also fits into the framework, as long as the exact solution stays close enough to the initial value.

4 Abstract evolution equation

Given a specific choice of η, and thereby a delay operator Φ, a possible setting for the problem at hand is the Banach space X given by

$$X = H \times L^p(-1,0;H;\tau),$$

where $1 \leq p < \infty$ determines the class of initial history segments that can be considered. The norm on X is given by

$$\|(u;\rho)\|_X = \Big(\|u\|_H^p + \int_{-1}^{0} \|\rho(\sigma)\|_H^p \tau(\sigma)\mathrm{d}\sigma\Big)^{1/p}.$$

For distributed delays (Assumption 2, Case 1) we can take the weight $\tau \equiv 1$. However, when e.g. considering point delays (Assumption 2, Case 2), $\tau \equiv 1$ does not yield dissipative operators [18, p. 76]. Instead, we let τ be the variation of η, i.e.

$$\tau(\sigma) = \int_{-1}^{\sigma} |\mathrm{d}\eta|.$$

We note that τ is a positive, increasing and bounded function. Because of the jump discontinuity in η at $\sigma = -1$ we have that $\lim_{\sigma\to -1} \tau(\sigma) > 0$, i.e. also $1/\tau$ is bounded. Finally, to move between the space X and its components we introduce the projections $P_1(u;\rho) = u$ and $P_2(u;\rho) = \rho$.

The operators F and G discussed in Section 2 can now be properly defined as

$$F = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 & g\Phi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix},$$

with the domains

$$\begin{aligned} &\mathscr{D}(F) = \mathscr{D}(f) \times L^p(-1,0;H;\tau), \\ &\mathscr{D}(G) = \big\{(u;\rho) \in X\,;\, \rho \in W^{1,p}(-1,0;H;\tau), u = \rho(0)\big\}, \end{aligned}$$

yielding the following abstract evolution equation on X:

$$\dot{U} = (F+G)U, \quad U(0) = (\zeta(0);\zeta). \tag{4.1}$$

We collect a few results regarding the operators F, G and $F+G$ before proceeding to the convergence analysis.

Lemma 4.1 *Under Assumption 1, the operator F is m-dissipative on X with $M[F] = 0$.*

Proof This follows by inspection.

As for G and $F+G$, Webb [18, Propositions 3.1, 3.2, 4.1 and 4.2] has proven the following:

Lemma 4.2 *Let Assumptions 1, 3 and 2 be satisfied. Then the operators G and $F+G$, with $\mathscr{D}(F+G) = \mathscr{D}(F)\cap\mathscr{D}(G)$, are both m-dissipative on X with*

$$M[G] = M[F+G] \leq M := \begin{cases} 1/p + \|\xi\|_{L^\infty(-1,0)}, & \textit{(Case 1)} \\ \tau(0), & \textit{(Case 2).} \end{cases}$$

The following lemma is a direct consequence of Definition 3.2:

Lemma 4.3 *If $E : \mathscr{D}(E) \subset X \to X$ is m-dissipative then the resolvent $(I-hE)^{-1} : X \to \mathscr{D}(E) \subset X$ is Lipschitz continuous with $L_X[(I-hE)^{-1}] \leq 1/(1-hM[E])$ for all $h \in (0, 1/M[E])$.*

Thus the time stepping operators $(I-hF)^{-1}(I-hG)^{-1}$ and $(I-h(F+G))^{-1}$ are both Lipschitz continuous. Further, there exists a unique mild solution U to the evolution equation (4.1) for every $U(0) \in \overline{\mathscr{D}(F+G)}$. The related solution operator is given by a nonlinear semigroup $\{S_{F+G}(t)\}_{t\geq 0}$, where $U(t) = S_{F+G}(t)U(0)$. The nonlinear operator $S_{F+G}(t)$ is invariant over the closure of $\mathscr{D}(F+G)$ and can be characterized by the limit

$$S_{F+G}(t)U(0) = \lim_{n\to\infty} \Big(I - \frac{t}{n}(F+G)\Big)^{-n} U(0).$$

See e.g. [1, Chapters 4] and [8] for proofs of these claims. Finally, we also know that the first component of $S_{F+G}(t)U(0)$ actually satisfies the original equation (3.1) for sufficiently smooth initial conditions, as per the following lemma [18, Proposition 5.8]:

Lemma 4.4 *Let Assumptions 1, 3 and 2 be valid. Further assume that $p \geq 2$, $\zeta(0) \in \mathscr{D}(f)$ and $\zeta \in W^{1,p}(-1,0;H;\tau)$. Then the function $u(t)$ given by*

$$u(t) = P_1 S_{F+G}(t)(\zeta(0);\zeta)$$

if $t \geq 0$ and $\zeta(t)$ if $t \in [-1,0)$ satisfies (3.1) *for almost all $t \geq 0$.*

5 Convergence

Let us introduce the following abbreviations for the time stepping operators related to the implicit Euler and Lie splitting schemes:

$$R_h = (I-h(F+G))^{-1}, \quad T_h = (I-hF)^{-1}(I-hG)^{-1}.$$

Given the initial history segment ζ, the schemes are then given by

$$(u_n;\rho_n) = R_h^n(\zeta(0);\zeta) \quad \text{and} \quad (v_n;\varphi_n) = T_h^n(\zeta(0);\zeta),$$

respectively, for $n = 0,1,2,\ldots$.

The work needed to state the delay differential equation as an abstract evolution equation on a specific space X now pays off, as we can apply the result of [8] to get a convergence order for the implicit Euler scheme. In the rest of this section we will make frequent use of the parameter M defined in Lemma 4.2, and C will denote a generic positive constant which assumes different values at different occurences.

Theorem 5.1 *Let Assumptions 1, 3 and 2 be satisfied and denote by $u(t)$ the solution to* (3.1). *Further, let the step size h satisfy $0 < hM \leq 1/2$. If $p \geq 2$, $\zeta \in W^{1,p}(-1,0;H;\tau)$ and $\zeta(0) \in \mathscr{D}(f)$, then*

$$\|u_n - u(nh)\|_H \leq Ch^{1/2}\|(F+G)(\zeta(0);\zeta)\|_X, \quad 0 \leq nh \leq T,$$

where the constant C depends on T but not on n or h.

Proof This follows from Lemma 4.4 and the proof of [8, Theorem 1].

We note that in many cases the order of convergence will actually be $q = 1$, for example when $H = \mathbb{R}^d$, but one can also find examples having only order $q = 1/2$; see e.g. [16]. There is currently no general framework for analysing when convergence orders $q > 1/2$ are to be expected.

The rest of this section is devoted to a similar theorem for the Lie splitting scheme.

Theorem 5.2 *Let Assumptions 1, 3 and 2 be satisfied and denote by $u(t)$ the solution to* (3.1). *Further, let the step size h satisfy $0 < hM \leq 1/2$. If $p \geq 2$, $\zeta \in W^{1,p}(-1,0;H;\tau)$ and $\zeta(0) \in \mathscr{D}(f)$, then*

$$\|v_n - u(nh)\|_H \leq C(h + h^{1/2} + h^{1-1/p})\big(\|(F+G)(\zeta(0);\zeta)\|_X + \|(\zeta(0);\zeta)\|_X\big),$$

where $0 \leq nh \leq T$ and the constant C depends on T but not on n or h.

Proof As

$$\|v_n - u(nh)\|_H \leq \|v_n - u_n\|_H + \|u_n - u(nh)\|_H,$$

where the second term can be bounded as in Theorem 5.1, the proof will be based on showing that the Lie splitting approximation is sufficiently close to the implicit Euler approximation. Even though we are only interested in the first component, the analysis needs to be performed in the full space. However, the norm with which we measure the history segments is of less importance. Alongside the space X we thus consider the space $Y = H \times L^1(-1,0;H;\tau)$, where

$$\|(u;\rho)\|_Y = \|u\|_H + \int_{-1}^{0} \|\rho(\sigma)\|_H \tau(\sigma)\mathrm{d}\sigma.$$

Clearly, $X \subset Y$ and $\|(u;\rho)\|_Y \leq C\|(u;\rho)\|_X$ for all $(u;\rho) \in X$. Further, due to Lemma 4.2 and Lemma 4.3 both $L_X[T_h]$ and $L_Y[T_h]$ are bounded by $1/(1-hM)$.

To shorten the notation we introduce $Z = (\zeta(0);\zeta) \in \mathscr{D}(F+G) \subset X$. Then

$$\|v_n - u_n\|_H = \|P_1\,(T_h^n Z - R_h^n Z)\|_H \leq \|R_h^n Z - T_h^n Z\|_Y.$$

Expanding the term $\|R_h^n Z - T_h^n Z\|_Y$ in a telescopic sum leads to

$$\begin{aligned}
\|R_h^n Z - T_h^n Z\|_Y &\leq \sum_{j=1}^{n} \|T_h^{n-j} R_h^j Z - T_h^{n-j+1} R_h^{j-1} Z\|_Y \\
&\leq \sum_{j=1}^{n} L_Y[T_h]^{n-j} L_Y[(I-hF)^{-1}] \|\big((I-hF)R_h - (I-hG)^{-1}\big) R_h^{j-1} Z\|_Y \\
&\leq C \sum_{j=1}^{n} \|\big((I-hF)R_h - (I-hG)^{-1}\big) R_h^{j-1} Z\|_Y,
\end{aligned}$$

where the last inequality follows as

$$L_Y[T_h]^j \le (1-hM)^{-j} \le \mathrm{e}^{2nhM} \tag{5.1}$$

when $hM \le 1/2$ for all $0 \le j \le n$. The operator $(I-hF)R_h - (I-hG)^{-1}$ is independent of j, and we can extract an h from it by the equality

$$\begin{aligned}(I-hF)R_h - (I-hG)^{-1} &= (hG + I - h(F+G))R_h - (I-hG)^{-1}\\ &= hGR_h + I - (I-hG)^{-1}\\ &= h(GR_h - G(I-hG)^{-1}).\end{aligned}$$

This h compensates for the n terms of the sum, hence any further powers of h which we can extract from $(GR_h - G(I-hG)^{-1})R_h^{j-1}$ yields the distance between the Lie splitting and implicit Euler approximations. We have, in fact, the following bound for any $U \in X$:

$$\|(GR_h - G(I-hG)^{-1})U\|_Y \le C\left(\|R_hU - U\|_X + (h + h^{1-1/p})\|U\|_X\right),$$

but since the proof is somewhat technical we defer this result to Lemma 5.1. Setting $U = R_h^{j-1}Z$ with $1 \le j \le n$ in the above inequality yields the terms

$$\begin{aligned}\|R_h^jZ - R_h^{j-1}Z\|_X &\le L_X[R_h^{j-1}]\|R_hZ - Z\|_X = L_X[R_h^{j-1}]\|R_hZ - R_h(I-h(F+G))Z\|_X\\ &\le hL_X[R_h]^j\|(F+G)Z\|_X \le h\mathrm{e}^{2nhM}\|(F+G)Z\|_X,\end{aligned}$$

by the same reasoning as in (5.1), and

$$\|R_h^{j-1}Z\|_X \le \|Z\|_X + \sum_{k=1}^{j-1}\|R_h^kZ - R_h^{k-1}Z\|_X \le \|Z\|_X + nh\mathrm{e}^{2nhM}\|(F+G)Z\|_X.$$

Thus, combining all the above inequalities finally yields

$$\begin{aligned}\|R_h^nZ - T_h^nZ\|_Y &\le Ch\sum_{j=1}^{n}\|\left(GR_h - G(I-hG)^{-1}\right)R_h^{j-1}Z\|_Y\\ &\le Ch\sum_{j=1}^{n}\|R_h^jZ - R_h^{j-1}Z\|_X + (h+h^{1-1/p})\|R_h^{j-1}Z\|_X\\ &\le Ch\sum_{j=1}^{n}h\|(F+G)Z\|_X + (h+h^{1-1/p})\left(\|(F+G)Z\|_X + \|Z\|_X\right)\\ &\le C(h+h^{1-1/p})\left(\|(F+G)Z\|_X + \|Z\|_X\right).\end{aligned}$$

This gives the desired bound. □

Lemma 5.1 *Let $U \in X$ and suppose that all the assumptions in Theorem 5.2 hold. Then, with Y as in Theorem 5.2,*

$$\|(GR_h - G(I-hG)^{-1})U\|_Y \le C\left(\|R_hU - U\|_X + (h + h^{1-1/p})\|U\|_X\right),$$

where C is independent of h.

Proof Assume that $U = (u;\rho) \in X$. Let $(v;\varphi) = (I-hG)^{-1}(u;\rho)$ and $(w;\psi) = R_h(u;\rho)$. Furthermore, let $z = k_h v + \Phi\theta_h$ with k_h and θ_h defined as in (2.3). Then by the discussion in Section 2 we have that

$$\begin{aligned}\varphi &= \sigma \mapsto e^{\sigma/h}(u+hgz)+\theta_h(\sigma) \quad \text{and}\\ \psi &= \sigma \mapsto e^{\sigma/h}w+\theta_h(\sigma).\end{aligned}$$

With this in place, we have the representation

$$\begin{aligned}P_1(GR_h - G(I-hG)^{-1})U &= g\Phi(\sigma \mapsto e^{\sigma/h}w+\theta_h(\sigma))\\ &\qquad - g\Phi(\sigma \mapsto e^{\sigma/h}(u+hgz)+\theta_h(\sigma)) \quad \text{and}\\ P_2(GR_h - G(I-hG)^{-1})U &= \sigma \mapsto \frac{1}{h}e^{\sigma/h}(w-u-hgz).\end{aligned}$$

Hence, we obtain the inequality

$$\begin{aligned}\|(GR_h - G(I-hG)^{-1})U\|_Y &\le L_H[g]\|\int_{-1}^{0} e^{\sigma/h}(w-u-hgz)\mathrm{d}\eta(\sigma)\|_H\\ &\quad + \int_{-1}^{0} \|\frac{1}{h}e^{\sigma/h}(w-u-hgz)\|_H \tau(\sigma)\mathrm{d}\sigma\\ &\le (L_H[g]|k_h| + r_h)\|w-u-hgz\|_H,\end{aligned}$$

where $|k_h|$ and

$$r_h = \int_{-1}^{0} \frac{1}{h}\mathrm{e}^{\sigma/h}\tau(\sigma)\mathrm{d}\sigma \le \tau(0)\int_{-1}^{0}\frac{1}{h}\mathrm{e}^{\sigma/h}\mathrm{d}\sigma = \tau(0)\left(1-\mathrm{e}^{-1/h}\right)$$

are both uniformly bounded by $\tau(0)$ for all $h > 0$. Now,

$$\begin{aligned}\|w-u-hgz\|_H &\le \|P_1(R_hU-U)\|_H + hL_H[g]\|z\|_H\\ &\le \|R_hU-U\|_X + hL_H[g]\Big(|k_h|\|P_1(I-hG)^{-1}U\|_H + \|\Phi\theta_h\|_H\Big)\\ &\le \|R_hU-U\|_X + hL_H[g]|k_h|L_X[(I-hG)^{-1}]\|U\|_X + hL_H[g]\|\Phi\theta_h\|_H,\end{aligned}$$

since $g(0)=0$ and $(I-hG)^{-1}0 = 0$. Further,

$$\begin{aligned}\|\Phi\theta_h\|_H &= \|\int_{-1}^{0}\int_{\sigma}^{0}\frac{1}{h}\mathrm{e}^{(\sigma-s)/h}\rho(s)\mathrm{d}s\,\mathrm{d}\eta(\sigma)\|_H\\ &\le \int_{-1}^{0}\int_{\sigma}^{0}\frac{1}{h}\mathrm{e}^{(\sigma-s)/h}\|\rho(s)\|_H\mathrm{d}s\,|\mathrm{d}\eta|(\sigma).\end{aligned}$$

We can bound the inner integral by Hölder's inequality. With $q > 1$ such that $1/q + 1/p = 1$ we have

$$\begin{aligned}\int_{\sigma}^{0}\frac{1}{h}\mathrm{e}^{(\sigma-s)/h}\|\rho(s)\|_H\mathrm{d}s &\le \frac{1}{h}\left(\int_{\sigma}^{0}\mathrm{e}^{q(\sigma-s)/h}\mathrm{d}s\right)^{1/q}\left(\int_{\sigma}^{0}\|\rho(s)\|_H^p\mathrm{d}s\right)^{1/p}\\ &\le \frac{1}{h}\left(\frac{h}{q}-\frac{h}{q}\mathrm{e}^{q\sigma/h}\right)^{1/q}\|\rho\|_{L^p(-1,0;H)}\\ &\le Ch^{1/q-1}\|\rho\|_{L^p(-1,0;H;\tau)}\\ &\le Ch^{-1/p}\|U\|_X,\end{aligned}$$

where we have used the fact that $1/\tau$ is bounded in the third inequality; see Section 4. Since $h^{-1/p}\|U\|_X$ does not depend on σ we obtain that

$$\|\Phi\theta_h\|_H \leq Ch^{-1/p}\|U\|_X.$$

Summarizing, we thus get

$$\|(GR_h - G(I-hG)^{-1})U\|_Y \leq C\left(\|R_hU - U\|_X + (h+h^{1-1/p})\|U\|_X\right),$$

which concludes the proof. □

6 Numerical experiments

In order to verify our results, we have performed two numerical experiments.

Example 6.1 Consider first the equation

$$\dot{u} = c_{\text{diff}}\Delta_r u + c_{\text{adv}}\frac{\mathrm{d}}{\mathrm{dx}}u + c_{\text{delay}}g\Phi u_t, \tag{6.1}$$

on the one-dimensional domain $\Omega = (0,1)$ with homogeneous Dirichlet boundary conditions. The operators g and Φ are given by $gu = u^2/(1+u^2)$ and

$$\Phi u_t = c_p u_t(-1) + c_d \int_{-1}^{0} u_t(\sigma)\mathrm{d}\sigma.$$

This resembles the population dynamics model (1.1) from the introduction, where a distributed delay has been added as well as an advection term, to make the solution more interesting. We choose the parameters $c_{\text{diff}} = 0.01$, $c_{\text{adv}} = 0.35$, $c_{\text{delay}} = 1.5$, $c_p = 0.1$ and $c_d = 4$. Note that the operator $f = \Delta_r + d/dx$ is m-dissipative and densely defined on $L^2(\Omega)$, see e.g. [19, Chapter 26], and clearly $g(u) = u^2/(1+u^2)$ is Lipschitz continuous. Further, $\eta(\sigma) = c_p\chi_{(-1,0]}(\sigma) + c_d(\sigma+1)$ yields the desired delay operator Φ. Assumptions 1, 3 and 2 (Case 2) are therefore valid.

We take $r = 9.5$ which makes the equation very stiff. We discretize the diffusion and advection operator with standard second order finite differences and $N_x = 501$ points in space. Since the methods are only of at most order $q = 1$ we represent the history segments ρ_n and φ_n by linear interpolation of N_h vectors, where we take N_h to be the same as the number of time steps on $[0,1]$. The initial condition is a scaled and shifted Barenblatt solution,

$$\zeta(0) = x \mapsto \left[1 - \frac{1}{17^{1/8.5}}|100(x-7/8)|^{9.5/8.5}\right]_+^{8.5/7.5}$$

and the initial history segment is given by $\zeta(\sigma) = \Lambda(\sigma)\zeta(0)$. Here $\Lambda(t)$ is 1 in the intervals $[-1,-0.9]$ and $[-0.5,-0.4]$, zero in $[-0.8,-0.6]$ and $[-0.3,-0.1]$ and connected linearly in between.

The integration is performed up to the final time $t = nh = 1.5$, using various numbers of time steps n in the range 2^6 to 2^{10}, and a reference solution is computed using the implicit Euler method with 2^{12} time steps. Figure 6.1 (left) shows the resulting

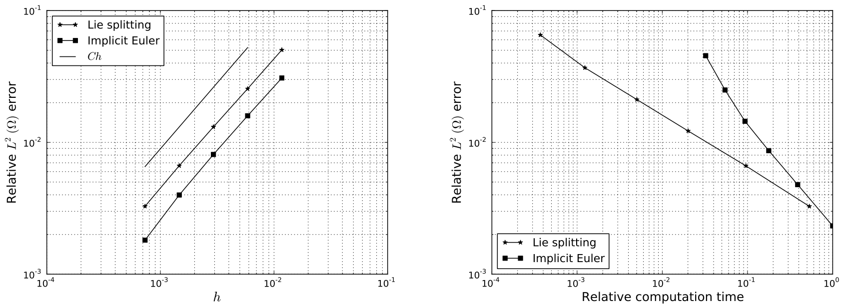

Fig. 6.1 Left: The $L^2(\Omega)$ errors corresponding to different step sizes h when approximating Equation (6.1) using implicit Euler or Lie splitting. We see that both methods are of order $q = 1$. Right: The $L^2(\Omega)$ errors when approximating (6.2) using implicit Euler or Lie splitting, plotted against the computation time. We see that the Lie splitting is more efficient than implicit Euler, with the gain depending on the accuracy level.

errors, measured in the discrete $L^2(\Omega)$ norm, for both the implicit Euler and the Lie splitting scheme. We observe that they are both of order $q = 1$, in line with Theorems 5.1 and 5.2. We also note that the errors are roughly of the same size, i.e. little accuracy is lost when employing the less costly splitting procedure.

Example 6.2 As a second example, consider the following semilinear equation:

$$\dot{u} = c_{\text{diff}} \Delta u + c_{\text{delay}} g \Phi u_t, \tag{6.2}$$

on the two-dimensional domain $\Omega = (0, 2\pi)^2$ with periodic boundary conditions. We let g and Φ have the same form as in Example 6.1 and choose the coefficients $c_{\text{diff}} = 1$, $c_{\text{delay}} = 1.5$, $c_p = 1$ and $c_d = 2$. For the implicit Euler method, we discretize the Laplacian by standard second-order finite differences and employ Newton iteration for step (A2:5). For the corresponding step (A1:5) of the Lie splitting scheme, we instead use fixed point iteration, and in step (A1:7) the action of $(I - hf)^{-1}$ is computed by FFT. In both cases we use $N_x = N_y = 256$ points in either space dimension, represent the history segments ρ_n and φ_n as in Example 6.1 and integrate up to $t = 1.5$. Finally, the initial condition is given by

$$\zeta(0)(x,y) = e^{-\left((x-\pi)^2 + (y-\pi)^2\right)},$$

and the initial history segment is given by $\zeta(\sigma) = \Lambda(\sigma)\zeta(0)$ with the same Λ as in Example 6.1.

The order of convergence q is again close to 1 for both methods. We omit the plot of this and consider instead the efficiency plot in Figure 6.1 (right). This shows the global errors (measured in the discrete $L^2(\Omega)$ norm) plotted against the computation time. We can see that for any given error level, the cost of the Lie splitting scheme is less than that of the implicit Euler method. For large step sizes, we see an improvement with a factor of $30 - 50$, while for smaller step sizes the cost approaches that of implicit Euler. This confluence is mainly due to the fact that for small step sizes the computation of θ_h is the dominating factor in both methods. The efficiency gain could be increased by further optimizing this part of the code.

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